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Path-integral analysis of scalar wave propagation in multiple-scattering random media

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In this work we consider the general problem of scalar wave propagation in a continuously inhomogeneous random medium, applying the approach originally proposed by Fock for the integration of quantum-mechanical equations. The principal idea of the method is based on the introduction of an additional pseudotime variable and the transfer to a higher-dimensional space, in which the propagation process is described by the generalized parabolic equation similar to the nonstationary Schrödinger equation in quantum mechanics. We present its solution in a form of Feynman path integral, the asymptotic evaluation of which in the far field allows us to estimate the so-called wave correction terms. These corrections are related to coherent backscattering and repeated multiple-scattering events on the same inhomogeneities, i.e., to the phenomena that are not described in the framework of the conventional theories of radiative transfer or small-angle scattering. As an example of the approach we consider the first statistical moments of the field for a point source located in a statistically homogeneous Gaussian random medium. The correction term obtained for the mean field coincides exactly with the classical result of the Bourret approximation for the Dyson equation, but with a much weaker restriction on the value of wave number k, which allows us to analyze the correction as a function of k. The main feature of the result obtained for the second moment of the field is that the normalized mean intensity is not equal to unity. We relate such behavior to the localization phenomenon. The dependence of the correction term does not differ significantly from that obtained in works concerning the localization of classical waves in discrete random media. [S1063-651X(96)06610-X]

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I. INTRODUCTION

The propagation of high-frequency radiation in random media has been the subject of investigation in various areas of physics for several decades. Some of the classical approaches are summarized in a number of monographs and review articles: see, e.g., [1–6]. In spite of its long history, the subject still presents a challenge when the complexity of propagation environments requires the development of new methods and the derivation of new solutions for the statistical measures of the field.

Most of the theories concerning the propagation of scalar time-harmonic waves are based on the reduced Helmholtz equation (HE). In a deterministic case a number of analytical and numerical techniques have been developed for solving this equation. However, in the presence of random inhomogeneities the HE acquires a stochastic character and direct multiple computations become impossible for most practically important situations.

In principle, the initial problem can be reformulated by dealing with integral equations for the mean field (Dyson equation) or for the first even statistical moments of the field (e.g. the Bethe-Salpeter equation for the coherence function) [1,5]. Unfortunately, the solutions of these equations can be obtained only by using some of the versions of perturbative techniques, for example, the Bourret approximation for the Dyson equation or the ladder approximation for the Bethe-Salpeter equation. These approximations reduce the problem to a phenomenological equation of radiative transfer in which some of the coherent effects are neglected [7]. At the same time it is the coherence and constructive interference between time-reversed multiply scattered waves that give rise to enhanced backscattering [8,9] and other double passage effects [10–12], related to the localization of classical waves [13,14], the phenomena that represent a topic of increasing current interest.

Of course, the information contained in the initial formulation, based on the scalar Helmholtz equation, is more complete and accounts for all the wave nature effects. However, the elliptic character of the HE with the resulting lack of the dynamic causality condition causes essential difficulties in the effective investigation of wave propagation in random media. In many situations the problem can be simplified by transfer to a parabolic-type equation (PE), which, in principle, allows one to account consistently for the wave nature of the propagation process even in the regime of strong fluctuations. The parabolic approximation is usually performed along some preferred geometrical ray of the background medium [15] and the resulting PE satisfies the causality condition, which makes it a suitable tool for solving stochastic problems. At the same time the approximations performed in its derivation restrict the standard parabolic equation (SPE) approach to small-angle scattering with backscatter fully neglected.

Another group of methods is based on the expansion of the unknown field into a series, each term of which is determined by the order of backscattering multiplicity. A comprehensive review of these methods can be found in [16]. However, the complexity of summing the multiple-scatter series

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strongly limits the applicability of this approach. Finally, the use of the invariant embedding method [17] reduces the problem to solving some equations of an evolutional type, but as a rule requires a special medium configuration.

In the present paper we adopt an approach that allows us also to reduce the initial formulation to a problem satisfying the causality condition and is related to the method originally proposed by Fock for the integration of quantum-mechanical equations [18]. The principal idea of the method is based on the introduction of an additional pseudotime variable and on the transfer to a higher-dimensional space in which the propagation process is described by the generalized parabolic equation (GPE) identical to the nonstationary Schrödinger equation in quantum mechanics [19]. The advantage of such a transfer, which can be viewed also as a version of an embedding technique, is that the solution of the resulting parabolic-type equation can be presented in a Feynman path-integral form [20,21]. In wave propagation theory the method was adopted by Buslaev [22], who converted the deterministic diffraction problem of asymptotic behavior for high-frequency radiation to that of solving the generalized diffusion (parabolic-type) equation. Frisch [1] applied this procedure to random propagation problems and presented a high-frequency approximation for the mean field using a Taylor expansion of the correlation function.

Further, this method, with some modifications, has been developed by many authors for both deterministic and stochastic wave propagation and scattering problems [23–27]. In particular, Klyatskin and Tatarskii [23] have used the method originally proposed by Fradkin in the quantum field theory to construct a path-integral solution for the field of a point source in a semi-infinite random medium. In the high-frequency limit, they estimated the corrections to the SPE solutions for the first two statistical moments of the field. Chow [24] has obtained the general expressions for the statistical moments approximating the ''classical action'' in the path integral by a quadratic functional around a stationary trajectory. Palmer [26] has applied the path-integral approach to the problem of underwater sound propagation and analyzed the stationary-phase approximation for the integral transform connecting the solution of the HE with that of the GPE. It has to be emphasized that the results presented in all the above-mentioned works are far from complete, primarily due to neglect of backscattering effects in the final expressions.

It is now well known that the path integral written in the configuration space can be approximately evaluated using an orthogonal expansion of each possible trajectory; the idea arises from Feynman's works [20]. For the SPE this method has been effectively applied to some problems of small-angle wave propagation in random media and described in detail in the recent review article of Charnotskii *et al.* [28]. We extend the approach developed for the SPE and present a solution for the generalized case. This allows us to calculate the so-called wave correction terms [7] and to establish sufficient conditions of applicability of the conventional SPE approach.

The outline of this work is as follows. First, in Sec. II we formulate the general equations describing time-harmonic wave propagation in nonhomogeneous media and consider the small-angle parabolic approximation. Then, in Sec. III we introduce the generalized parabolic equation and present its solution in a path-integral form. In order to reduce the continual integral to its finite N-dimensional version we expand each virtual path into an eigenfunction series, similarly to what was done for the SPE [28]. While being similar from the mathematical point of view, these two solutions, however, are characterized by considerably different physical contents since the extension performed contains trajectories describing Nth-order backscattering events. The unknown solution of the HE is related to the GPE solution by a special integral transform. In the far field this relation contains a highly oscillatory function, which makes extremely difficult its direct numerical evaluation. To simplify the problem, in Sec. IV, following Palmer [26], we present the solution of the HE in the form of a series, the first term of which is the solution of the GPE and accounts for the main contribution to the unknown field. The following terms represent a series in GPE solution derivatives, which allows us to evaluate the corrections.

To exemplify the proposed approach we consider in Sec. V the mean field of a point source in a statistically homogeneous Gaussian random medium. Using a perturbative technique and representing the unknown function as a sum of a leading term plus a correction, we obtain some asymptotic expressions for both large- and small-scale inhomogeneities of the medium. The second-order coherence function and higher statistical moments of the wave field are analyzed in Sec. VI. Further, in Sec. VII we apply the results obtained to the asymptotic analysis of classical wave localization in random media. Section VIII of the paper contains a summary and some principal concluding remarks.

II. FORMULATION OF THE PROBLEM

We start with the time-harmonic Helmholtz equation describing the propagation and scattering of scalar waves in nonhomogeneous media. For the field of a point source located at $\mathbf{R}_0$ this equation has the form

$$\nabla^2 G(\mathbf{R}|\mathbf{R}_0)+k^2[1+\tilde{\varepsilon}(\mathbf{R})]G(\mathbf{R}|\mathbf{R}_0)=-\delta(\mathbf{R}-\mathbf{R}_0), \quad (2.1)$$

where $\mathbf{R}$ denotes the position vector in m-dimensional space ($m=2$ or 3), k is a wave number of the homogeneous medium, and $\varepsilon(\mathbf{R})=1+\tilde{\varepsilon}(\mathbf{R})$ is the permittivity distribution, in which $\tilde{\varepsilon}(\mathbf{R})$ is the random perturbation. We suppose that while ε is a real function, k contains an infinitesimally small positive imaginary part ($\mathrm{Im}\, k>0$) that provides the convergence of some integrals appearing in the course of the work.

Equation (2.1) is equivalent to the Lippmann-Schwinger integral equation

$$G(\mathbf{R}|\mathbf{R}_0)=G_0(\mathbf{R}|\mathbf{R}_0) + k^2\int d^mR'\,\tilde{\varepsilon}(\mathbf{R}')G_0(\mathbf{R}'|\mathbf{R}_0)G(\mathbf{R}|\mathbf{R}'), \quad (2.2)$$

where $G_0(\mathbf{R}|\mathbf{R}_0)$ is the free-space Green's function satisfying the equation

$$\nabla^2 G_0(\mathbf{R}|\mathbf{R}_0)+k^2G_0(\mathbf{R}|\mathbf{R}_0)=-\delta(\mathbf{R}-\mathbf{R}_0) \quad (2.3)$$

and radiation condition at infinity. In the m-dimensional space the Green's function $G_0(\mathbf{R}|\mathbf{R}_0)$ is given by

$$G_0(\mathbf{R}|\mathbf{R}_0)=(i/4)(k/2\pi|\mathbf{R}-\mathbf{R}_0|)^{m/2-1}H^{(1)}_{m/2-1}(k|\mathbf{R}-\mathbf{R}_0|). \tag{2.4}$$

In the far field $k|\mathbf{R}-\mathbf{R}_0|\gg 1$ one can use the first term of the asymptotic expansion of the Hankel function to approximate $G_0(\mathbf{R}|\mathbf{R}_0)$ by the formula

$$G_0(\mathbf{R}|\mathbf{R}_0)=\tfrac{1}{2}(i/k)^{(3-m)/2}(2\pi|\mathbf{R}-\mathbf{R}_0|)^{(1-m)/2}\times\exp(ik|\mathbf{R}-\mathbf{R}_0|), \tag{2.5}$$

which is exact for $m=3$.

In a random medium the functions of interest are the statistical measures such as mean field $\langle G(\mathbf{R}|\mathbf{R}_0)\rangle$ or second-order coherence function $\langle G(\mathbf{R}_1|\mathbf{R}_{01})G(\mathbf{R}_2|\mathbf{R}_{02})\rangle$ (the angular brackets denote ensemble averaging). In principle, these measures may be obtained by solving the Dyson or Bethe-Salpeter equations, respectively [5]. An alternative approach is based on the presentation of an unknown solution of Eq. (2.1) as a function of random perturbation $\tilde{\varepsilon}(\mathbf{R})$. However, the lack of the so-called dynamic causality condition due to the elliptic character of Eq. (2.1) causes essential difficulties when one attempts to obtain such solutions. Physically it means that, as a rule, it is not possible to find the direction of spatial movement for which the field values at each subsequent point would be determined by the parameters of the medium at the preceding spatial locations only. Therefore it is desirable to convert the initial problem to some auxiliary evolutional-type formulation, which would satisfy the causality condition [17].

The conventional method that realizes this idea is based on the transfer from Eq. (2.1) to the approximate parabolic-type equation describing the small-angle scattering [5]. Extracting the main phase term, we denote the reduced wave function $g(\mathbf{r},z|\mathbf{r}_0,z_0)$ by the relation

$$G(\mathbf{R}|\mathbf{R}_0)=\exp[ik(z-z_0)]g(\mathbf{r},z|\mathbf{r}_0,z_0), \tag{2.6}$$

where the z axis corresponds to a preferred propagation direction of the wave in a nonperturbed medium (range coordinate) and $\mathbf{r}$ is an $(m-1)$-dimensional vector in the transverse plane (cross-range coordinate). Neglecting the second range derivative yields the standard form of the parabolic equation

$$2ik\partial_z g+\nabla^2_{\mathbf{r}}+k^2\tilde{\varepsilon}(\mathbf{r},z)g(\mathbf{r},z|\mathbf{r}_0,z_0)=0, \tag{2.7a}$$

with the initial condition

$$g(\mathbf{r},z_0|\mathbf{r}_0,z_0)=\delta(\mathbf{r}-\mathbf{r}_0). \tag{2.7b}$$

Unlike G, the field g satisfies the causality condition in a sense that the value $g(\mathbf{r},z|\mathbf{r}_0,z_0)$ depends functionally upon previous values of $\tilde{\varepsilon}(\mathbf{r},z)$ only, i.e., on inhomogeneities in the layer $z_0\leqslant z'<z$. Moreover, the SPE coincides formally with the nonstationary Schrödinger equation in the $(m-1)$-dimensional space and therefore one can apply, by analogy with the latter, the path-integral approach, which has been already demonstrated its productivity in various cases [21,28].

However, the transfer to the SPE is justified only if the inhomogeneities of the medium are sufficiently weak, smooth, and large scale (compared to the wavelength), so that the propagation process is localized in the paraxial zone. In practice there are many important problems permitting this approach, but, nevertheless, there are arbitrary situations for which some of the above assumptions are not fulfilled. In such situations it is necessary to bring into consideration a theory that could satisfy the causality condition and, at the same time, account for the multiple scattering including the backscattering effects. To this end we transfer to the equation that we refer as generalized parabolic one.

III. GENERALIZED PARABOLIC EQUATION

Let us consider an auxiliary problem for a function $\tilde{G}(\mathbf{R},\tau|\mathbf{R}_0,\tau_0)$ satisfying the equation

$$2ik\partial_\tau\tilde{G}+\nabla^2\tilde{G}+k^2[1+\tilde{\varepsilon}(\mathbf{R})]\tilde{G}(\mathbf{R},\tau|\mathbf{R}_0,\tau_0)=0,\quad \tau>\tau_0, \tag{3.1a}$$

with initial condition

$$\tilde{G}(\mathbf{R},\tau_0|\mathbf{R}_0,\tau_0)=\delta(\mathbf{R}-\mathbf{R}_0). \tag{3.1b}$$

It is implied also that the function $\tilde{G}$ satisfies the radiation condition, i.e., it vanishes for $\mathbf{R}\to\infty$ or $\tau\to\infty$ (a small absorption Im $k>0$ is taken into account; for Im $k=0$ the required solution is obtained by an analytical continuation [1,22]). Comparing Eq. (3.1) with (2.1), it can be shown [18] that their solutions are related by the integral transform

$$G(\mathbf{R}|\mathbf{R}_0)=\frac{i}{2k}\int_{\tau_0}^{\infty}d\tau\ \tilde{G}(\mathbf{R},\tau|\mathbf{R}_0,\tau_0). \tag{3.2}$$

Defining the function $\tilde{G}$ as

$$\tilde{G}(\mathbf{R},\tau|\mathbf{R}_0,\tau_0)=\exp\left[i\,\frac{k}{2}\,(\tau-\tau_0)\right]g(\mathbf{R},\tau|\mathbf{R}_0,\tau_0) \tag{3.3}$$

and substituting it into Eq. (3.1), we obtain the generalized parabolic equation

$$2ik\partial_\tau g+\nabla^2 g+k^2\tilde{\varepsilon}(\mathbf{R})g(\mathbf{R},\tau|\mathbf{R}_0,\tau_0)=0,\quad \tau>\tau_0 \tag{3.4a}$$

$$g(\mathbf{R},\tau_0|\mathbf{R}_0,\tau_0)=\delta(\mathbf{R}-\mathbf{R}_0). \tag{3.4b}$$

Hence the Green's function $G(\mathbf{R}|\mathbf{R}_0)$ is defined through the solution of the latter equation as

$$G(\mathbf{R}|\mathbf{R}_0)=\frac{i}{2k}\int_{\tau_0}^{\infty}d\tau\ \exp\left[i\,\frac{k}{2}\,(\tau-\tau_0)\right]g(\mathbf{R},\tau|\mathbf{R}_0,\tau_0). \tag{3.5}$$

We note that the generalized parabolic equation (3.4) has a higher dimension than the classical one.

For homogeneous medium ($\tilde{\varepsilon}=0$) the solution of Eq. (3.4) is given by

$$g_0(\mathbf{R},\tau|\mathbf{R}_0,\tau_0)=\left[\frac{k}{2\pi i(\tau-\tau_0)}\right]^{m/2}\exp\left[\frac{ik(\mathbf{R}-\mathbf{R}_0)^2}{2(\tau-\tau_0)}\right]. \tag{3.6}$$

Substituting Eq. (3.6) into (3.5) and taking into account a known integral representation of the Hankel function [29], we obtain the required expression (2.4) for the free-space Green's function of the Helmholtz equation.

The generalized parabolic equation (3.4) for the Green's function $g(\mathbf{R},\tau|\mathbf{R}_0,\tau_0)$ coincides with the nonstationary Schrödinger equation in quantum mechanics. Using this analogy, the solution of the GPE can be presented in the Feynman path-integral form

$$g(\mathbf{R},\tau|\mathbf{R}_0,\tau_0)=\int_{\mathbf{R}(\tau_0)=\mathbf{R}_0}^{\mathbf{R}(\tau)=\mathbf{R}} D\mathbf{R}(t)\exp\{iS[\mathbf{R}(t)]\}, \quad (3.7)$$

where the integration $\int D\mathbf{R}(t)$ in the continuum of possible trajectories is interpreted as a sum of contributions of arbitrary paths along which the wave propagates from point $\mathbf{R}_0$ at the moment τ_0 to point $\mathbf{R}$ at the moment τ and the functional

$$S[\mathbf{R}(t)]=\frac{k}{2}\int_{\tau_0}^{\tau} dt\{[\dot{\mathbf{R}}(t)]^2+\tilde{\varepsilon}[\mathbf{R}(t)]\} \quad (3.8)$$

can be related to the phase accumulated along the corresponding path. The measure $D\mathbf{R}(t)$ in Eq. (3.7) is chosen so that the normalization condition for a homogeneous medium

$$\int_{\mathbf{R}(\tau_0)=\mathbf{R}_0}^{\mathbf{R}(\tau)=\mathbf{R}} D\mathbf{R}(t)\exp\left\{i\,\frac{k}{2}\int_{\tau_0}^{\tau} dt[\dot{\mathbf{R}}(t)]^2\right\}=g_0(\mathbf{R},\tau|\mathbf{R}_0,\tau_0) \quad (3.9)$$

is satisfied.

The path integral can be exactly evaluated only for Gaussian integrands, i.e., for the functionals $S[\mathbf{R}(t)]$ of a quadratic type. Investigation of disordered media requires an approximate computation of the path integral. Using the approach initially proposed in quantum mechanics [20,28], we expand each virtual trajectory into the series

$$\mathbf{R}(t)=\overline{\mathbf{R}}(t)+\sum_{n=1}^{N}\psi_n(t)\mathbf{Q}_n, \quad (3.10)$$

where

$$\overline{\mathbf{R}}(t)=\frac{\tau-t}{\tau-\tau_0}\,\mathbf{R}_0+\frac{t-\tau_0}{\tau-\tau_0}\,\mathbf{R} \quad (3.11)$$

is a straight line connecting the points $\mathbf{R}_0$ and $\mathbf{R}$ and $\psi_n(t)$ is a complete set of orthogonal functions, e.g.,

$$\psi_n(t)=\frac{\sqrt{2(\tau-\tau_0)}}{\pi n}\sin\left(\frac{\pi n t}{\tau-\tau_0}\right). \quad (3.12)$$

As a result, the path integral can be presented as a product

$$g(\mathbf{R},\tau|\mathbf{R}_0,\tau_0)=g_0(\mathbf{R},\tau|\mathbf{R}_0,\tau_0)g_\varepsilon(\mathbf{R},\tau|\mathbf{R}_0,\tau_0), \quad (3.13)$$

where g_0 is the free-space Green's function (3.6) and the inhomogeneous factor g_ε is a limit $N\to\infty$ of the following finite-dimensional approximation (hereafter we use the notation $\underline{N}=1,2,\ldots,N$):

$$g_\varepsilon(\mathbf{R},\tau|\mathbf{R}_0,\tau_0)=\left(\frac{k}{2\pi i}\right)^{mN/2}\int d^{mN}Q_{\underline{N}}\exp\left(i\,\frac{k}{2}\sum_{n=1}^{N}Q_n^2\right)\times\exp\left\{i\,\frac{k}{2}\int_{\tau_0}^{\tau}dt\;\tilde{\varepsilon}\left[\overline{\mathbf{R}}(t)+\sum_{n=1}^{N}\psi_n(t)\mathbf{Q}_n\right]\right\}. \quad (3.14)$$

Formally, the path-integral representations for the standard and generalized parabolic equations are identical, except that the latter has a dimension higher by unity than the first one. However, this difference has a significant physical content. As was already noted, the parabolic equations have a causal character. This means that the SPE describes the scattering process in the forward direction only, accounting for the trajectories, which do not have any turning point with respect to the range coordinate z. The same restriction for the GPE takes place with respect to the auxiliary pseudotime coordinate τ. If we consider the projection $\mathbf{R}_\tau(t)$ of a consequent path onto the real m-dimensional space, we find that the generalized formulation allows trajectories with multiple (Nth-order) turning points. Hence, while the number of terms taken into account in the series expansion (3.10) for the SPE determines only the accuracy of the results, for the GPE this number has an additional physical interpretation as the backscattering multiplicity. For this reason, representation (3.14) may be considered as an analog of the series expansion discussed in [16].

Direct numerical evaluation of the exact formula (3.5) is extremely difficult since in the far field the integrand contains a highly oscillatory function. To simplify the calculations, in the next section we derive a series expansion of the integral transform (3.5).

IV. SERIES EXPANSION FOR THE GREEN'S FUNCTION

As shown in the preceding section, the Green's function of the GPE can be presented as a product of two factors, the first corresponding to the free space and the second related to the spatial fluctuations $\tilde{\varepsilon}(\mathbf{R})$. Substituting representation (3.13) into Eq. (3.5) gives

$$G(\mathbf{R}|\mathbf{R}_0)=\frac{i}{2k}\int_{\tau_0}^{\infty}d\tau\exp\left[i\,\frac{k}{2}(\tau-\tau_0)\right]\times g_0(\mathbf{R},\tau|\mathbf{R}_0,\tau_0)g_\varepsilon(\mathbf{R},\tau|\mathbf{R}_0,\tau_0). \quad (4.1)$$

Next, applying a formal identity

$$g_\varepsilon(\mathbf{R},\tau|\mathbf{R}_0,\tau_0)=\int_{-\infty}^{\infty}ds\;\delta(s-\tau)g_\varepsilon(\mathbf{R},s|\mathbf{R}_0,\tau_0), \quad (4.2)$$

representing the δ function by its spectral expansion, and interchanging integration order allows us to evaluate the integral over τ. This leads to the expression

$$G(\mathbf{R}|\mathbf{R}_0)=\int_{-\infty}^{\infty}ds\;g_\varepsilon(\mathbf{R},s|\mathbf{R}_0,\tau_0)\times\frac{1}{2\pi}\int_{-\infty}^{\infty}d\Omega\;\exp[i(s-\tau_0-L)\Omega]F_m(\Omega), \quad (4.3)$$

where the function $F_m(\Omega)$ is defined as

$$F_m(\Omega)=(i/4)(k\sqrt{1-2\Omega/k}/2\pi L)^{m/2-1}\exp(iL\Omega)\times H^{(1)}_{m/2-1}(k\sqrt{1-2\Omega/k}L) \tag{4.4}$$

and $L=|\mathbf{R}-\mathbf{R}_0|$ is the distance between two given points in the m-dimensional configurational space. The function $\exp(-iL\Omega)F_m(\Omega)$ is the free-space Green's function for the wave with a wave number equal to $k\sqrt{1-2\Omega/k}$. Alternatively, we can write down Eq. (4.3) as

$$G(\mathbf{R}|\mathbf{R}_0)=\int_{-\infty}^{\infty}ds\ f_m(s)g_\varepsilon(\mathbf{R},\tau_0+L-s|\mathbf{R}_0,\tau_0), \tag{4.5}$$

where the function $f_m(s)$ is the inverse Fourier transform

$$f_m(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\Omega\ \exp(-is\Omega)F_m(\Omega). \tag{4.6}$$

Representing the function $F_m(\Omega)$ in Eq. (4.3) by a Taylor series in the neighborhood of $\Omega=0$, we obtain the following series expansion for the unknown propagator:

$$G(\mathbf{R}|\mathbf{R}_0)=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}F_m^{(n)}(0)g_\varepsilon^{(n)}(\mathbf{R},\tau_0+L|\mathbf{R}_0,\tau_0). \tag{4.7}$$

The coefficients $F_m^{(n)}(0)$ may be simply evaluated for the far field approximation (2.5), which allows us to present the Green's function as a product of two factors

$$G(\mathbf{R}|\mathbf{R}_0)=G_0(\mathbf{R}|\mathbf{R}_0)G_\varepsilon(\mathbf{R}|\mathbf{R}_0), \tag{4.8}$$

i.e., in a form similar to Eq. (3.13). The inhomogeneous factor $G_\varepsilon(\mathbf{R}|\mathbf{R}_0)$ in the three-dimensional case is given by the following series in derivatives of the GPE solution:

$$\begin{aligned}G_\varepsilon(\mathbf{R}|\mathbf{R}_0)&=g_\varepsilon(\mathbf{R},\tau_0+L|\mathbf{R}_0,\tau_0)\\&\quad+i(L/2k)g_\varepsilon''(\mathbf{R},\tau_0+L|\mathbf{R}_0,\tau_0)\\&\quad+(L/2k^2)g_\varepsilon'''(\mathbf{R},\tau_0+L|\mathbf{R}_0,\tau_0)+\cdots .\end{aligned} \tag{4.9}$$

Formally, the result is similar to the asymptotic expansion of DeSanto's integral transform, relating the solutions of the Helmholtz equation and the SPE in an arbitrary two-dimensional waveguide [30,31]. The principal difference is that Eq. (4.9) contains the solution of the GPE instead of the SPE.

Keeping only the first term in this series and neglecting all the derivatives, we obtain

$$G_\varepsilon(\mathbf{R}|\mathbf{R}_0)\approx g_\varepsilon(\mathbf{R},\tau_0+L|\mathbf{R}_0,\tau_0). \tag{4.10}$$

The applicability limits for this approximation may be evaluated by applying Rytov's complex phase approach, specifically representing the g_ε factor as

$$g_\varepsilon(\mathbf{R},\tau_0+L|\mathbf{R}_0,\tau_0)=\exp[\varphi_\varepsilon(\mathbf{R},\tau_0+L|\mathbf{R}_0,\tau_0)], \tag{4.11}$$

where φ_ε is the component of the complex phase related to the inhomogeneities effects. Then the series (4.9) reduces to the multiplicative form

$$G_\varepsilon(\mathbf{R}|\mathbf{R}_0)=g_\varepsilon(\mathbf{R},\tau_0+L|\mathbf{R}_0,\tau_0)\{1+i(L/2k)[\varphi_\varepsilon'^2+\varphi_\varepsilon'']+(L/2k^2)[\varphi_\varepsilon'^3+3\varphi_\varepsilon'\varphi_\varepsilon''+\varphi_\varepsilon''']+\cdots\}. \tag{4.12}$$

Obviously Eq. (4.10) is satisfied if all the additional terms in Eq. (4.12) are small. Physically it means that, for instance, the changes of the complex phase φ_ε of the GPE solution on a distance $l_F=\sqrt{L/k}$ (transverse size of the first Fresnel zone) must be small compared to unity. This condition seems to state rigid limitations for many deterministic problems, but as we will see below it can be essentially relaxed for the statistical moments of the wavefield propagating in random media.

V. MEAN FIELD

In this section we shall investigate the abilities of the proposed approach for obtaining an approximate solution for the mean field, combining the series expansion for the Green's function and the path integration of the GPE. According to Eqs. (4.8) and (4.9) the mean field radiated by a point source is given by

$$\begin{aligned}\langle G(\mathbf{R}|\mathbf{R}_0)\rangle&=G_0(\mathbf{R}|\mathbf{R}_0)\{\langle g_\varepsilon(\mathbf{R},\tau_0+L|\mathbf{R}_0,\tau_0)\rangle\\&\quad+i(L/2k)\langle g_\varepsilon''(\mathbf{R},\tau_0+L|\mathbf{R}_0,\tau_0)\rangle+\cdots\}.\end{aligned} \tag{5.1}$$

We assume that the fluctuations of the medium are described by Gaussian statistics. Furthermore, let the random medium be statistically homogeneous, i.e., let its correlation function

$$B_\varepsilon(\mathbf{R})=\langle\tilde\varepsilon(\mathbf{R}_1)\tilde\varepsilon(\mathbf{R}_2)\rangle \tag{5.2}$$

depend only on the difference vector $\mathbf{R}=\mathbf{R}_1-\mathbf{R}_2$. In this case, using Eq. (3.14), we find that the first term in expansion (5.1) has the form

$$\begin{aligned}\langle g_\varepsilon(\mathbf{R},\tau_0+L|\mathbf{R}_0,\tau_0)\rangle&=\left(\frac{k}{2\pi i}\right)^{mN/2}\int d^{mN}Q_{\underline{N}}\ \exp\left(i\ \frac{k}{2}\sum_{n=1}^{N}Q_n^2\right)\\&\quad\times\exp\left\{-\frac{k^2}{8}\int_0^L dt_1\int_0^L dt_2F_1(t_1,t_2;\mathbf{Q}_{\underline{N}})\right\},\end{aligned} \tag{5.3}$$

where the scattering function $F_1(t_1,t_2;\mathbf{Q}_{\underline{N}})$ is given by

$$F_1(t_1,t_2;\mathbf{Q}_{\underline{N}})=B_\varepsilon\left(\mathbf{T}+\sum_{n=1}^{N}[\psi_n(t_1)-\psi_n(t_2)]\mathbf{Q}_n\right) \tag{5.4}$$

and the vector $\mathbf{T}$ is defined as $\mathbf{T}=\overline{\mathbf{R}}(t_1)-\overline{\mathbf{R}}(t_2)$. For dimensionless variables the first exponent in Eq. (5.3) contains the parameter kL, which is much greater than unity in the far field. Therefore, asymptotic evaluation of this integral, as well as of similar integrals appearing further in the analysis of the coherence function and higher-order statistical moments of the field, can be based upon application of Erdelyi's lemma [32]

$$\int_0^\infty dx\, x^{\nu-1} \exp(isx^\eta) f(x)$$

$$\sim \sum_{n=0}^\infty \frac{\Gamma[(n+\nu)/\eta]}{n!\,\eta} (-is)^{-(n+\nu)/\eta} f^{(n)}(0), \quad |s|\to\infty. \tag{5.5}$$

So the values of statistical moments of the field are prescribed by the behavior of the corresponding integrand for small values of $\mathbf{Q}_{\underline{N}}$. The leading term of the series expansion for the statistically isotropic correlation function $B_\varepsilon(R)$, after introducing new sum and difference variables

$$t' = (1/2)(t_1+t_2), \quad t = t_1 - t_2, \tag{5.6}$$

and integrating over t', is given by

$$\langle \overline{g_\varepsilon}(\mathbf{R},\tau_0+L|\mathbf{R}_0,\tau_0)\rangle = \exp\left\{-\frac{k^2}{4}\int_0^L dt(L-t)B_\varepsilon(t)\right\}. \tag{5.7}$$

This formula, originally obtained by Klyatskin and Tatarskii, represents the simplest generalization of the Markov approximation applied to the standard parabolic equation [5]. This term is purely real and describes the extinction of the coherent part of the field.

As an example we shall perform the calculations for the isotropic correlation function of a Gaussian form with characteristic correlation scale l_ε:

$$B_\varepsilon(R) = \sigma_\varepsilon^2 \exp(-R^2/l_\varepsilon^2). \tag{5.8}$$

The leading term in this case is given by

$$\langle \overline{g_\varepsilon}(\mathbf{R},\tau_0+L|\mathbf{R}_0,\tau_0)\rangle = \exp(-\alpha L), \tag{5.9}$$

where α is the extinction coefficient. If the normalized path length $\ell = L/l_\varepsilon$ is much greater than unity, we have

$$\alpha \approx \alpha_0 = (\sqrt{\pi}/8)k^2 l_\varepsilon \sigma_\varepsilon^2. \tag{5.10}$$

In order to calculate the correction to (5.9) we present the mean-field solution in the form

$$\begin{aligned}\langle g_\varepsilon(\mathbf{R},\tau_0+L|\mathbf{R}_0,\tau_0)\rangle &= \langle \overline{g_\varepsilon}(\mathbf{R},\tau_0+L|\mathbf{R}_0,\tau_0)\rangle \\ &\times \left(\frac{k}{2\pi i}\right)^{mN/2} \int d^{mN}Q_{\underline{N}} \exp\left(i\,\frac{k}{2}\sum_{n=1}^N Q_n^2\right) \\ &\times \exp\left\{\frac{k^2}{8}\int_0^L dt_1 \int_0^L dt_2 \widetilde{F}_1(t_1,t_2;\mathbf{Q}_{\underline{N}})\right\},\end{aligned} \tag{5.11}$$

where

$$\widetilde{F}_1(t_1,t_2;\mathbf{Q}_{\underline{N}}) = F_1(t_1,t_2,0) - F_1(t_1,t_2;\mathbf{Q}_{\underline{N}}). \tag{5.12}$$

For small values of $\mathbf{Q}_{\underline{N}}$ we can expand the second exponent in Eq. (5.11) and represent the solution as

$$\langle g_\varepsilon(\mathbf{R},\tau_0+L|\mathbf{R}_0,\tau_0)\rangle = \langle \overline{g_\varepsilon}(\mathbf{R},\tau_0+L|\mathbf{R}_0,\tau_0)\rangle\{1+\delta+\cdots\}. \tag{5.13}$$

Using the spectral representation of the correlation function

$$B_\varepsilon(\mathbf{R}) = \int d^mK \exp(i\mathbf{R}\cdot\mathbf{K})\Phi_\varepsilon(\mathbf{K}), \tag{5.14}$$

we obtain for the first correction term δ in both two- and three-dimensional cases the expression

$$\delta = \frac{k^2}{8}\int_0^L dt_1 \int_0^L dt_2 \int d^mK \Phi_\varepsilon(\mathbf{K})\exp(i\mathbf{T}\cdot\mathbf{K}) \times[1-\exp(-i\eta K^2)], \tag{5.15}$$

where

$$\eta(N) = \frac{1}{2k}\sum_{n=1}^N [\psi_n(t_1)-\psi_n(t_2)]^2. \tag{5.16}$$

An exact summation for $N\to\infty$ leads to

$$\eta = (L/2k)(t/L)(1-t/L), \tag{5.17}$$

where t is defined by the second of Eqs. (5.6). Slow convergence to the exact result ($N\to\infty$) is observed. The same conclusion was made in [21] regarding analogous asymptotic expansions for the scintillation index evaluated in the framework of the SPE. From the mathematical point of view this is caused by the difference in the velocity of growth of the exact function η and its finite version $\eta(N)$. Unlike the approximation of η by several first terms, the exact function is linear for small t. Such a distinction can provide not only a quantitative difference in the solution but also dramatic changes in its behavior. The sufficient number of the eigenfunctions is proportional to the value of ℓ. Therefore, an adequate result may be obtained only by taking into account a large number of eigenfunctions i.e., a high multiplicity of backscattering events.

Since, besides that, the exact version of η depends on the difference coordinate t only, we can perform easily one more integration with the result

$$\delta = \frac{k^2}{4}\int_0^L dt(L-t)\int d^mK \Phi_\varepsilon(\mathbf{K})\exp(i\mathbf{T}\cdot\mathbf{K}) \times[1-\exp(-i\eta K^2)]. \tag{5.18}$$

For $\eta\to 0$ we can approximate the last exponent in Eq. (5.18) by the first two terms of its series expansion. Such a replacement is correct only if the transverse size of the first Fresnel zone is much smaller than the minimal scale of random inhomogeneities l_ε, i.e., in the domain of applicability of geometrical optics. In this case δ is purely imaginary. Evidently, the imaginary component of δ is related to the mean phase shift.

For example, in three dimensions the correction term is defined by

$$\delta = i(\pi/2)kL\int_0^\infty dK\, K^2\{1-2(LK)^{-2} \times[1-\cos(LK)]\}\Phi_\varepsilon(K). \quad (5.19)$$

The Gaussian correlation function (5.8) corresponds to the spectral density of random inhomogeneities ($m=3$)

$$\Phi_\varepsilon(K) = (1/8\pi^{3/2})\sigma_\varepsilon^2 l_\varepsilon^3 \exp(-l_\varepsilon^2 K^2/4). \quad (5.20)$$

Using this spectrum and performing the integration in Eq. (5.19) leads to

$$\delta = i\tfrac{1}{8}\kappa\ell\{1-\ell^{-2}[1-\exp(-\ell^2)]\}\sigma_\varepsilon^2, \quad (5.21)$$

where $\kappa = kl_\varepsilon$ is the normalized wave number.

For the opposite situation, when $L\to\infty$ such as $L/kl_\varepsilon^2\to\infty$ we obtain the expression

$$\mathrm{Im}\ \delta = (\pi/2)k^2L\int_0^\infty dK\, K\ \ln|(2k+K)/(2k-K)|\Phi_\varepsilon(K), \quad (5.22)$$

which for the same Gaussian spectrum can be presented as

$$\mathrm{Im}\ \delta = (\sqrt{\pi}/8)c(\kappa)\kappa^2\ell\sigma_\varepsilon^2, \quad (5.23)$$

with the coefficient

$$c(\kappa) = (2/\pi)\kappa^2\int_0^\infty dz\ z\ \exp(-\kappa^2z^2)\ln|(1+z)/(1-z)|. \quad (5.24)$$

This result coincides exactly with the corresponding formula obtained by solving the Dyson equation in the case of small-scale inhomogeneities ($\kappa\ll 1$) [5]. Our derivation is based on the requirement $\kappa/\ell\ll 1$, which is much weaker for large values of ℓ and allows us to analyze the behavior of Im δ as a function of κ. Comparing Eq. (5.23) with the extinction $\alpha_0 L = (\sqrt{\pi}/8)\kappa^2\ell\sigma_\varepsilon^2$, we note that the coefficient $c(\kappa)$ defines approximately the value of mean phase shift normalized to the extinction of the coherent part of the field. Graphically the variation $c(\kappa)$ is shown in Fig. 1. We see that the dependence has a resonant structure with the maximum located at $\kappa\sim 1$.

Now we shall estimate the next term in the series expansion (5.1). Taking into account only the leading term and interchanging the order of statistical averaging and differentiation, we find for the correction

$$\delta' = -i(1-2\alpha L)\exp(-\alpha L)\alpha/k. \quad (5.25)$$

This correction is equal to the extinction on the wavelength scale. It is clear that this term can be significant only in the geometrical optics regime. Finally, it is useful to note that our estimates of all the correction terms give the sufficient conditions of applicability of the classical formulation based on the Markov approximation for the SPE and define more precisely the necessary conditions obtained in [4,5].

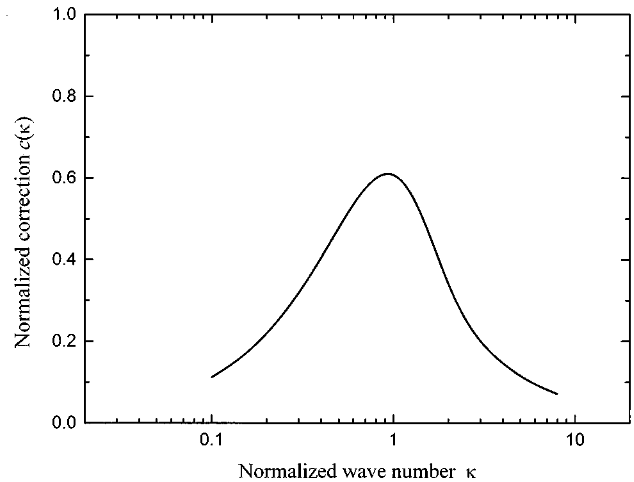

FIG. 1. Normalized correction to the mean field for $\kappa/\ell\ll 1$ in three dimensions.

VI. COHERENCE FUNCTION AND HIGHER-ORDER MOMENTS

Using the analysis of the mean field as a test for the approximations performed, we can now move to the evaluation of the coherence function and higher-order statistical moments, which could be helpful for the description of the statistical properties of the field propagating in strongly inhomogeneous media. For the point source located at the origin $\mathbf{R}_0=0$, the normalized (with free space factor removed) coherence function is defined as

$$\gamma_2(\mathbf{R}_1,\mathbf{R}_2) = \langle G_\varepsilon(\mathbf{R}_1|\mathbf{R}_0)G_\varepsilon^*(\mathbf{R}_2|\mathbf{R}_0)\rangle. \quad (6.1)$$

Choosing the observation points $\mathbf{R}_1$ and $\mathbf{R}_2$ to be located on a sphere centered on a point source location and using the series expansion in the form of Eq. (4.9), we can also write down a similar expression for the coherence function

$$\begin{aligned}\gamma_2(\mathbf{R}_1,\mathbf{R}_2) = {}& \langle g_\varepsilon(\mathbf{R}_1,\tau_0+L|\mathbf{R}_0,\tau_0)g_\varepsilon^*(\mathbf{R}_2,\tau_0+L|\mathbf{R}_0,\tau_0)\rangle \\ &+(L/2k)^2\langle g_\varepsilon''(\mathbf{R}_1,\tau_0+L|\mathbf{R}_0,\tau_0) \\ &\times g_\varepsilon''^*(\mathbf{R}_2,\tau_0+L|\mathbf{R}_0,\tau_0)\rangle \\ &+(L/k^2)\langle g_\varepsilon(\mathbf{R}_1,\tau_0+L|\mathbf{R}_0,\tau_0) \\ &\times g_\varepsilon'''^*(\mathbf{R}_2,\tau_0+L|\mathbf{R}_0,\tau_0)\rangle + \cdots , \end{aligned} \quad (6.2)$$

where L is the radius of the observation sphere. Similarly to the case of the mean field, it can be shown that in the far field $kL\gg 1$, the coherence function is well defined by keeping only the first term of the corresponding series expansion. Let us introduce the angle θ between the observation points. Hence, performing averaging we find

$$\gamma_2(\theta,L)=(k/2\pi)^{mN}\int d^{mN}P_{\underline{N}}\int d^{mN}Q_{\underline{N}}\times\exp\left(ik\sum_{n=1}^{N}\mathbf{P}_n\cdot\mathbf{Q}_n\right)\exp\left[-\frac{k^2}{4}\int_0^L dt_1\times\int_0^L dt_2F_2(t_1,t_2;\mathbf{P}_{\underline{N}},\mathbf{Q}_{\underline{N}})\right], \tag{6.3}$$

where the scattering function F_2 is defined by

$$F_2(t_1,t_2;\mathbf{P}_{\underline{N}},\mathbf{Q}_{\underline{N}})=D_\varepsilon[\mathbf{R}_1(t_1)-\mathbf{R}_2(t_2)]-\frac{1}{2}\sum_{j=1}^{2}D_\varepsilon[\mathbf{R}_j(t_1)-\mathbf{R}_j(t_2)] \tag{6.4}$$

and D_ε is the structure function of the inhomogeneities [5]

$$D_\varepsilon(\mathbf{R}_1,\mathbf{R}_2)=\langle[\tilde{\varepsilon}(\mathbf{R}_1)-\tilde{\varepsilon}(\mathbf{R}_2)]^2\rangle. \tag{6.5}$$

For a statistically homogeneous medium the structure function depends only on the difference vector and is related to the correlation function as

$$D_\varepsilon(\mathbf{R})=2[B_\varepsilon(0)-B_\varepsilon(\mathbf{R})]. \tag{6.6}$$

The vectors

$$\mathbf{R}_j(t)=\overline{\mathbf{R}_j}(t)+\sum_{n=1}^{N}\psi_n(t)[\mathbf{P}_n+(-1)^{j-1}\mathbf{Q}_n/2],\quad j=1,2 \tag{6.7}$$

in Eq. (6.4) define the trajectories connecting the source with the corresponding observation point.

Applying Erdeyi's lemma to Eq. (6.3), we find that the leading term is given by

$$\overline{\gamma_2}(\theta,L)=\exp\left[-\frac{k^2}{4}\int_0^L dt_1\int_0^L dt_2F_2(t_1,t_2;0,0)\right]. \tag{6.8}$$

As the mean field, we can represent the coherence function in the form of a series expansion

$$\gamma_2(\theta,L)=\overline{\gamma_2}(\theta,L)\{1+\chi+\cdots\}, \tag{6.9}$$

where the first correction χ is presented by the expression

$$\chi=\frac{k^2}{4}\int_0^L dt_1\int_0^L dt_2(k/2\pi)^{mN}\int d^{mN}P_{\underline{N}}\int d^{mN}Q_{\underline{N}}\times\exp\left(ik\sum_{n=1}^{N}\mathbf{P}_n\cdot\mathbf{Q}_n\right)\tilde{F}_2(t_1,t_2;\mathbf{P}_{\underline{N}},\mathbf{Q}_{\underline{N}}), \tag{6.10}$$

in which

$$\tilde{F}_2(t_1,t_2;\mathbf{P}_{\underline{N}},\mathbf{Q}_{\underline{N}})=F_2(t_1,t_2;0,0)-F_2(t_1,t_2;\mathbf{P}_{\underline{N}};\mathbf{Q}_{\underline{N}}). \tag{6.11}$$

Next, using the spectral form of the structure function

$$D_\varepsilon(\mathbf{R})=2\int d^mK[1-\exp(i\mathbf{R}\cdot\mathbf{K})]\Phi_\varepsilon(\mathbf{K}), \tag{6.12}$$

we obtain

$$\chi=\frac{k^2}{2}\int_0^L dt_1\int_0^L dt_2\int d^mK\Phi_\varepsilon(\mathbf{K})\{\exp(i\mathbf{T}_1\cdot\mathbf{K})\times[1-\cos(\eta K^2)]-\exp(i\mathbf{T}_2\cdot\mathbf{K})[1-\cos(\tilde{\eta}K^2)]\}, \tag{6.13}$$

where

$$\mathbf{T}_1=\overline{\mathbf{R}_j}(t_1)-\overline{\mathbf{R}_j}(t_2),\quad \mathbf{T}_2=\overline{\mathbf{R}_1}(t_1)-\overline{\mathbf{R}_2}(t_2), \tag{6.14}$$

and

$$\tilde{\eta}(N)=\frac{1}{2k}\sum_{n=1}^{N}[\psi_n^2(t_1)-\psi_n^2(t_2)]. \tag{6.15}$$

Performing exact summation in Eq. (6.15) for $N\to\infty$ leads to

$$\tilde{\eta}=(L/2k)(t/L)(1-2t'/L), \tag{6.16}$$

where t and t' are defined by Eqs. (5.6).

Applying the same approximation, all higher-order statistical moments of the field can be evaluated. For example, the second-order intensity moment (fourth-order moment of the field) is given as

$$\gamma_4(\theta,L)=\left(\frac{k}{2\pi}\right)^{2mN}\int d^{4mN}Q_{\underline{4}\,\underline{N}}\times\exp\left[i\frac{k}{2}\sum_{n=1}^{N}\sum_{j=1}^{4}(-1)^{j-1}Q_{jn}^2\right]\times\exp\left[-\frac{k^2}{4}\int_0^L dt_1\int_0^L dt_2F_4(t_1,t_2;\mathbf{Q}_{\underline{4}\,\underline{N}})\right], \tag{6.17}$$

where

$$F_4(t_1,t_2;\mathbf{Q}_{\underline{4}\,\underline{N}})=\frac{1}{2}\sum_{j=1}^{4}D_\varepsilon[\mathbf{R}_j(t_1)-\mathbf{R}_j(t_2)]+D_\varepsilon[\mathbf{R}_1(t_1)-\mathbf{R}_3(t_2)]+D_\varepsilon[\mathbf{R}_2(t_1)-\mathbf{R}_4(t_2)]-D_\varepsilon[\mathbf{R}_1(t_1)-\mathbf{R}_2(t_2)]-D_\varepsilon[\mathbf{R}_1(t_1)-\mathbf{R}_4(t_2)]-D_\varepsilon[\mathbf{R}_2(t_1)-\mathbf{R}_3(t_2)]-D_\varepsilon[\mathbf{R}_3(t_1)-\mathbf{R}_4(t_2)], \tag{6.18}$$

and the trajectories $\mathbf{R}_j(t)$ are defined as

$$\mathbf{R}_j(t)=\overline{\mathbf{R}_s}(t)+\sum_{n=1}^{N}\psi_n(t)\mathbf{Q}_{jn},\quad j=1,\ldots,4. \tag{6.19}$$

Here $s=1$ for $j=1,2$ and $s=2$ for $j=3,4$. Using the appropriate manipulations, we can also apply a similar integration procedure for the fourth moment as well as for other higher-order statistical moments. However, computations of the resulting expressions in the general case require the use of special versions of the Monte Carlo method [33–35]. We hope that an analytical asymptotic solution can be obtained in the strong fluctuations regime and we intend to analyze this elsewhere in the future.

VII. TRANSITION TO LOCALIZATION

The normalized mean intensity $\iota(L)$ can be obtained from the coherence function $\gamma_2(\theta,L)$ for $\theta=0$:

$$\iota(L)=1+\chi+\cdots, \tag{7.1}$$

where

$$\chi=\frac{k^2}{2}\int_0^L dt_1\int_0^L dt_2\int d^mK\ \Phi_\varepsilon(\mathbf{K})\cos(\mathbf{T}\cdot\mathbf{K})\times[\cos(\tilde{\eta}K^2)-\cos(\eta K^2)]. \tag{7.2}$$

We see that, according to Eqs. (7.1) and (7.2), $\iota(L)$ is not equal to unity in the general case. Obviously such behavior can be recognized as a manifestation of the so-called weak localization of classical waves [14]. We expect that investigation of this asymptotic result can be helpful in the description of the transition from extended to localized states.

Introducing in Eq. (7.2) the new variables, defined by (5.6), and integrating over the difference coordinate t, we obtain

$$\chi=k^2\int_0^L dt(L-t)\int d^mK\ \Phi_\varepsilon(\mathbf{K})\cos(\mathbf{T}\cdot\mathbf{K})\times[(\eta K^2)^{-1}\sin(\eta K^2)-\cos(\eta K^2)]. \tag{7.3}$$

For the geometrical optics regime this expression in the three-dimensional case reduces to

$$\chi=2\pi\int_0^\infty dK\ K^2\{1+(LK)^{-1}\sin(LK)-4(LK)^{-2}\times[1-\cos(LK)]\}\Phi_\epsilon(K). \tag{7.4}$$

Using the Gaussian spectrum (5.20), we find that

$$\chi=\tfrac{1}{2}[1-2\ell^{-2}+(1+2\ell^{-2})\exp(-\ell^2)]\sigma_\varepsilon^2. \tag{7.5}$$

For the opposite situation when $L\to\infty$ and $L/kl_\varepsilon^2\to\infty$ we have the asymptotic formula

$$\chi=4\pi^2k^3L\int_{2k}^\infty dK\ \Phi_\varepsilon(K). \tag{7.6}$$

Performing the integration in Eq. (7.6) with the same spectrum leads to

$$\chi=s(\kappa)\ell\sigma_\varepsilon^2, \tag{7.7}$$

where

$$s(\kappa)=(\pi/2)\kappa^3\ \mathrm{erfc}(\kappa). \tag{7.8}$$

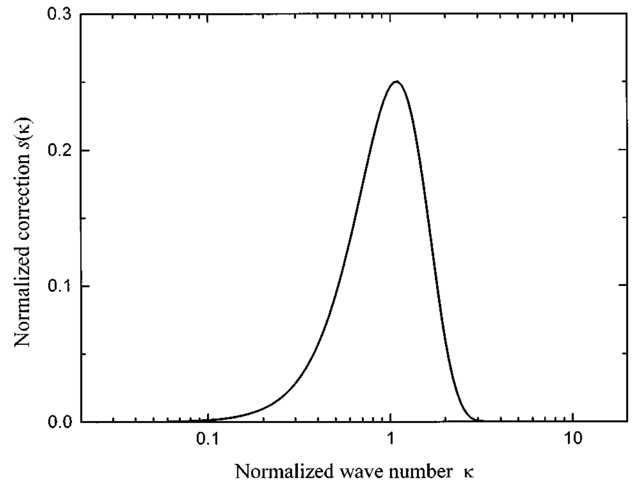

FIG. 2. Normalized correction to the mean intensity for $\kappa/\ell\ll 1$ in three dimensions.

The dependence of the coefficient s on the normalized wave number κ is shown in Fig. 2. We see that there is a sufficiently narrow window of κ for which an essential increase of the correction can be observed. The wavelength in this window is comparable to the correlation length l_ε and this is the natural estimate in the framework of the wave localization concept [14].

VIII. SUMMARY AND CONCLUSIONS

In this paper, we have implemented the Feynman path-integral approach to the problems of scalar wave propagation in random media. For this purpose we have applied the method originally proposed by Fock for the integration of quantum-mechanical equations. The principal idea of the method is based on the introduction of an additional pseudotime variable and the transfer to a higher-dimensional space in which the propagation process is described by a generalized parabolic equation similar to the nonstationary Schrödinger equation in quantum mechanics. The advantage of such a transfer, which can be interpreted also as a version of the embedding technique, is that the solution of the parabolic-type equation can be presented in a path-integral form. Such a representation, which also allows us to construct the statistical moments of the field, may be considered as a general solution of the problem, accounting for all the wave-nature effects. The question, however, is how to transform this general mathematical structure into a physically tractable form. To do this we first represented the path integral by a limit $N\to\infty$ to a finite N-dimensional integral over the expansion coefficients of each virtual path into an eigenfunction series. The number of terms taken into account in the series expansion has a clear physical interpretation as the backscattering multiplicity. Furthermore, the representation has the multiplicative form, namely, the form of products of two factors, one of which is related to the homogeneous medium, while the other reflects the effect of the inhomogeneities. Presenting the unknown propagator $G(\mathbf{R}|\mathbf{R}_0)$ in a similar multiplicative form, we restricted our attention to the far-field approximation, for which the corresponding inhomogeneous factor is given by that of GPE.

As an example, the application of the proposed approach is examined for the first statistical moments of the field excited by a point source in a statistically homogeneous Gaussian random medium. In the Markov approximation for the random fluctuations of the medium the result reduces to the known solution of the standard parabolic equation. To obtain a more correct estimate, we applied a perturbative technique, presenting the solution as a sum of a leading term, which is merely the SPE solution, and the correction term, which accounts for the coherent backscattering effects. Analyzing the results obtained as functions of the number of eigenfunctions, we found that even for the simplest perturbative approach applied here, the results contain much more information than the approaches accounting for a small number of backscattering events.

The correction term obtained for the mean field coincides exactly with the classical result of the Bourret approximation for the Dyson equation but with a much weaker restriction on the value of the wave number k, which allows us to analyze the correction as a function of k. We have shown that the dependence has a resonant structure with the maximum corresponding to a wavelength having the order of magnitude of the characteristic scale of the inhomogeneous medium.

A similar analysis may also be used for the asymptotic evaluation of the coherence function and higher-order moments. Here we have presented quite general expressions for both second and fourth moments of the field. In the case of the second-order coherence function a simple asymptotic formula was derived. The main feature of the result obtained is that in the far field the normalized mean intensity is not equal to unity. We connect such behavior with the localization phenomenon, specifically with the asymptotic transition from extended to localized states. In fact, as was shown, the dependence of the correction term on the wave number has a quite narrow peak centered at the typical spatial frequency in the spectral density of the medium. This dependence does not differ significantly from that obtained in classical works concerning wave localization in discrete random media [14].

Hence the technique applied in our work allows us to account for the coherent backscattering effects and to evaluate the wave correction terms for all statistical moments of the field. On the other hand, the corrections obtained permit us to establish the sufficient condition of applicability of the standard parabolic equation in various situations. We also hope that the limitations of the approach, due to the asymptotic procedures used, could be essentially relaxed by application of direct numerical techniques, particularly Monte Carlo methods, to evaluation of the functional integrals.

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